Resource Booklet – Round 2

ATM/MA London 2017

UCL/IOE, Bedford Way, London

UNIVERSITY OF CAMBRIDGE

Department for Education

Contents

Making connections

Mathematics is a coherent and connected enterprise. To reflect this we have organised our resources along a system of thematic tube lines.

These resources support teachers in the classroom. They help students build firm foundations for mathematical understanding by connecting ideas and developing techniques.
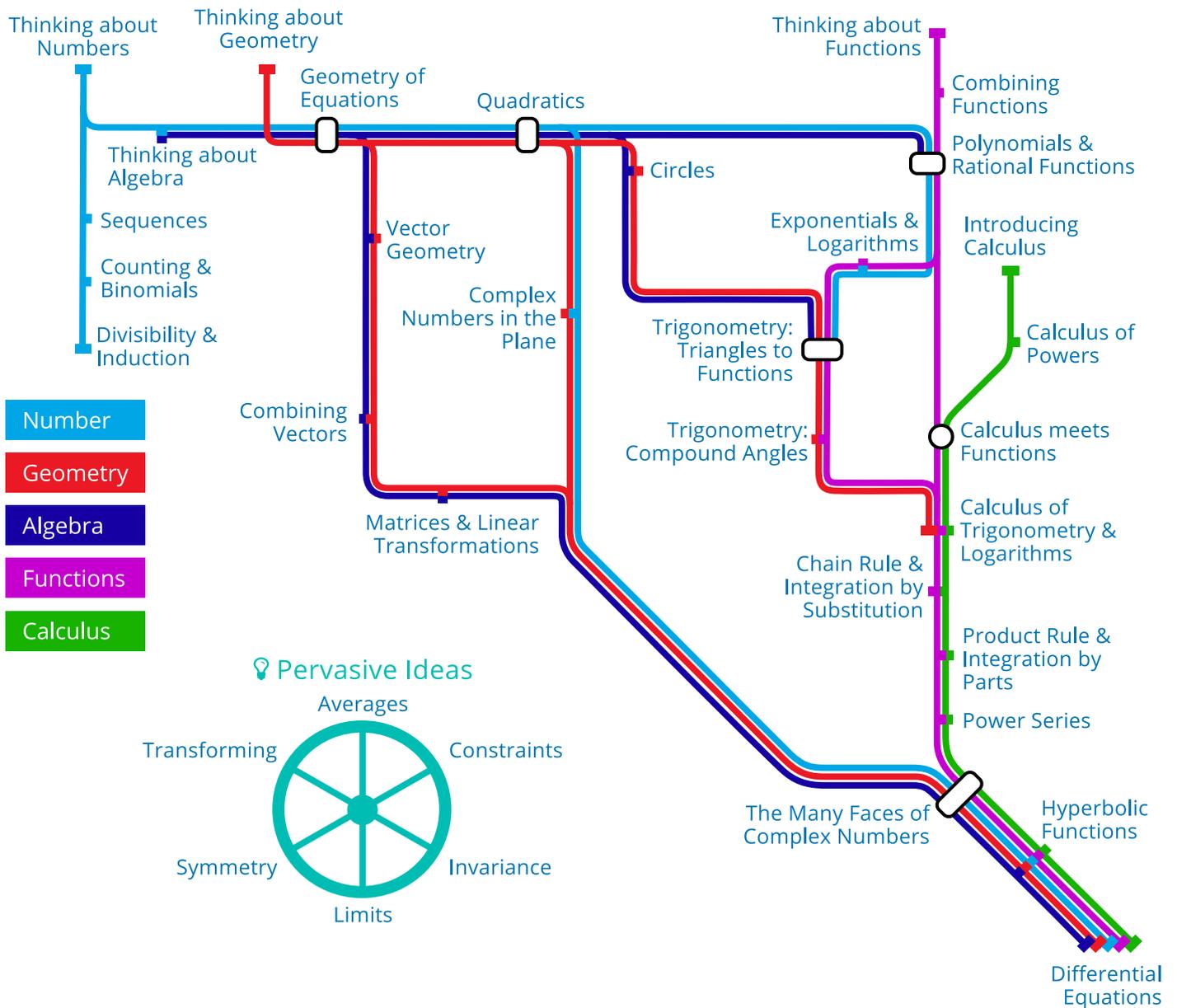
The resources are designed to be used in such a way as to

- give students the opportunity to think mathematically;
- support students in developing their own understanding;
- invite students to make connections for themselves;
- nurture students' mathematical independence, and
- help students to develop resilience, flexibility and creativity.

This new approach to post-16 mathematics is being developed by the University of Cambridge, funded by a grant from the UK Department for Education. The resources are free for all users.

Underground Mathematics started in 2012 as the Cambridge Mathematics Education Project (CMEP).

underground mathematics

Exploring the connections that underpin mathematics

Thinking about Numbers

Thinking about Geometry

Geometry of Equations

Quadratics

Thinking about Functions

Combining Functions

Thinking about Algebra

Polynomials & Rational Functions

Circles

Sequences

Vector Geometry

Exponentials & Logarithms

Introducing Calculus

Counting & Binomials

Complex Numbers in the Plane

Divisibility & Induction

Calculus of Powers

Trigonometry: Triangles to Functions

Calculus meets Functions

Combining Vectors

Trigonometry: Compound Angles

Calculus of Trigonometry & Logarithms

Number

Geometry

Algebra

Functions

Calculus

Matrices & Linear Transformations

Chain Rule & Integration by Substitution

Product Rule & Integration by Parts

Power Series

💡 Pervasive Ideas

Averages

Transforming

Constraints

Symmetry

Invariance

The Many Faces of Complex Numbers

Hyperbolic Functions

Limits

Differential Equations

Near miss

Problem

Solve the two pairs of simultaneous equations

$$x + 0.99999y = 2.99999$$
$$0.99999x + y = 2.99998$$

and

$$x + 1.00001y = 2.99999$$
$$0.99999x + y = 2.99998.$$

Were you surprised by your answers?

Explain why the solutions are so different and yet the pairs of equations are nearly identical.

Last updated 04-Nov-15

Pick a card...

Quadratics of the form $f(x) = x^2 + bx + c$

underground
mathematics

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①

$$f(x) = \ldots$$

(Function in form $x^2 + bx + c$)

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②

Graph of $y = f(x)$

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③

The graph crosses the axes at $x = \ldots\ldots$, $x = \ldots\ldots$ and $y = \ldots\ldots$.

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④

$$f(0) = \ldots$$

$$f(1) = \ldots$$

$$f(2) = \ldots$$

---

⑤

$$f(x) = (x \ldots\ldots\ldots)^2 \ldots\ldots\ldots$$

(Function in completed square form)

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⑥

The lowest point on the graph is $(\ldots\ldots, \ldots\ldots)$.

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⑦

x	-3	-2	-1	0	1	2	3
y							

---

⑧

The solution(s) of $f(x) = 0$ is/are ...

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⑨

$$f(x) = (\ldots\ldots\ldots)(\ldots\ldots\ldots)$$

(Function in fully factorised form)

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https://undergroundmathematics.org/quadratics/pick-a-card

Discriminating

Problem

Below are several statements about the quadratic equation $ax^2 + bx + c = 0$, where a, b and c are allowed to be any real numbers except that a is not 0.

For each statement, decide whether MUST, MAY or CAN'T is the correct word to use in the statement.

☞
To say that something MUST be the case, we need it to be true in *all* cases; we will need to give a convincing explanation (a proof) of why this must be always true.

To show that something CAN'T be the case, we likewise need to give a convincing explanation (a proof) of why.

To show that something MAY be the case, we need to give an example when it is true and an example when it is false. If you want a harder challenge, can you determine exactly when it is and when it is not true?

(1) If $a < 0$, then $ax^2 + bx + c = 0$ MUST / MAY / CAN'T have real roots.

(2) If $b^2 - 4ac = 0$, then $ax^2 + bx + c = 0$ MUST / MAY / CAN'T have one repeated real root.

(3) If $ax^2 + bx + c = 0$ has no real roots, then $ax^2 + bx - c = 0$ MUST / MAY / CAN'T have two distinct real roots.

(4) If $\frac{b^2}{a} < 4c$, then $ax^2 + bx + c = 0$ MUST / MAY / CAN'T have two distinct real roots.

(5) If $b = 0$, then $ax^2 + bx + c = 0$ MUST / MAY / CAN'T have one repeated real root.

(6) The equation $ax^2 + bx + c = 0$ MUST / MAY / CAN'T have three real roots.

(7) If $c = 0$, then $ax^2 + bx + c = 0$ MUST / MAY / CAN'T have real roots.

(8) The equation $ax^2 + bx + c = 0$ MUST / MAY / CAN'T have the same number of real roots as $ax^2 - bx + c = 0$.

(9) If $ax^2 + bx + c = 0$ has two distinct real roots, then we MUST / MAY / CAN'T have $ac < \frac{b^2}{4}$.

(10) If $c > 0$, then $ax^2 + bx + c = 0$ MUST / MAY / CAN'T have two distinct real roots.

(11) The equation $ax^2 + bx + c = 0$ MUST / MAY / CAN'T have the same number of real roots as $cx^2 + bx + a = 0$.

(12) If $ax^2 + bx + c = 0$ has no real roots, then $-ax^2 - bx - c = 0$ MUST / MAY / CAN'T have two distinct real roots.

Last updated 06-Jan-17

This shows that the original equation is equivalent to

$$\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a} = 0.$$

Since $a \neq 0$, we can divide by a to get

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0.$$

We complete the square.

We can rewrite the right-hand side by putting it over a common denominator:

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}.$$

Consider $ax^2 + bx + c = 0$, where $a \neq 0$.

Get the squared term on one side of the equation:

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a}.$$

Subtracting $\frac{b}{2a}$ from both sides and putting the right-hand side over a common denominator gives

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

We can take the square root of both sides.

Since x appears only once in the equation, we can rearrange this to solve for x.

Taking account of the possibility of positive and negative square roots, we see

$$x + \frac{b}{2a} = \frac{\pm\sqrt{b^2 - 4ac}}{2a}.$$
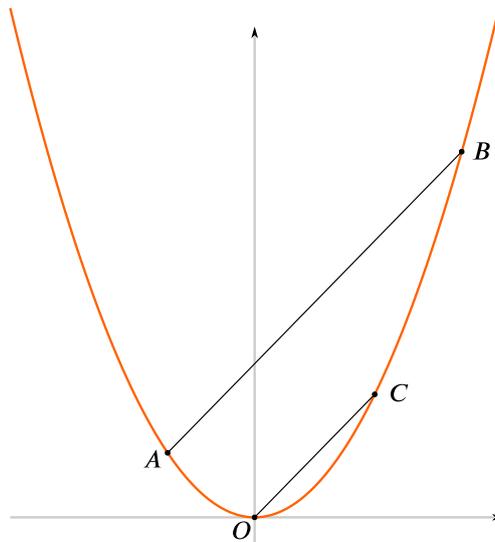
Parabella

Problem

Take any two points A and B on the parabola $y = x^2$.

Draw the line OC through the origin, parallel to AB, cutting the parabola again at C.

Let A have coordinates (a, a^2), let B have coordinates (b, b^2) and let C have coordinates (c, c^2).

Prove that $a + b = c$.

The parabella

Imagine drawing another parallel line DE, where D and E are two other points on the parabola. Extend the ideas of the previous result to prove that the midpoints of each of the three parallel lines lie on a straight line.

Is the Serpentine Lake really 40 acres?

Serpentine map

Review question R6641

Question

Draw a diagram showing points $A(-3, 1)$, $B(4, 2)$, $C(9, -3)$, $D(2, -4)$.

(i) For the figure $ABCD$ state and prove a fact (or facts) sufficient to ensure that it is a parallelogram.

(ii) State and prove an additional fact sufficient to ensure that $ABCD$ is a rhombus.

(iii) Prove that $ABCD$ is not a square.

Review question R6735

Question

A function f is defined by $f : x \to \dfrac{1}{x+1}$. Write down in similar form expressions for f^{-1} and ff.

It is required to find the values of x for which (i) $f = f^{-1}$, (ii) $f = ff$. Show that, in each case, the values of x are given by the equation

$$x^2 + x - 1 = 0.$$

Review question R6993

Question

The fact that

$$6 \times 7 = 42,$$

is a counter-example to which of the following statements?

(a) the product of any two odd integers is odd;

(b) if the product of two integers is not a multiple of 4 then the integers are not consecutive;

(c) if the product of two integers is a multiple of 4 then the integers are not consecutive;

(d) any even integer can be written as the product of two even integers.

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Rich resources for teaching A level mathematics

Enabling all students to explore the connections that underpin mathematics

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